A systematic regularised method with its application in magnetised matter

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Outline

- Dimension Reduction
- Summation of All Landau Levels
- Lerch-Zeta Function
- Vacuum Polarization
- Conclusions

Magnetic Fields

- $1\ \mathrm{MeV}^2 = 1.44 \times 10^{13}$ Gauss, $m_\pi^2 \sim 2.8 \times 10^{17}$ Gauss
- In heavy ion collisions: 10^{18} to 10^{19} Gauss $(\Delta t \sim 10^{-24} s)$
- Compact stars: 10^{10} to 10^{15} Gauss
- Early Universe: up to 10^{24} Gauss

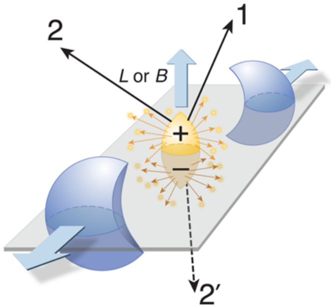


Illustration by Carin Cain

Landau Levels

- Free energy spectrum

$$E_n^{3+1}(p_3) = \pm\sqrt{(2n + 2s_3 + 1)|eB| + p_3^2}$$

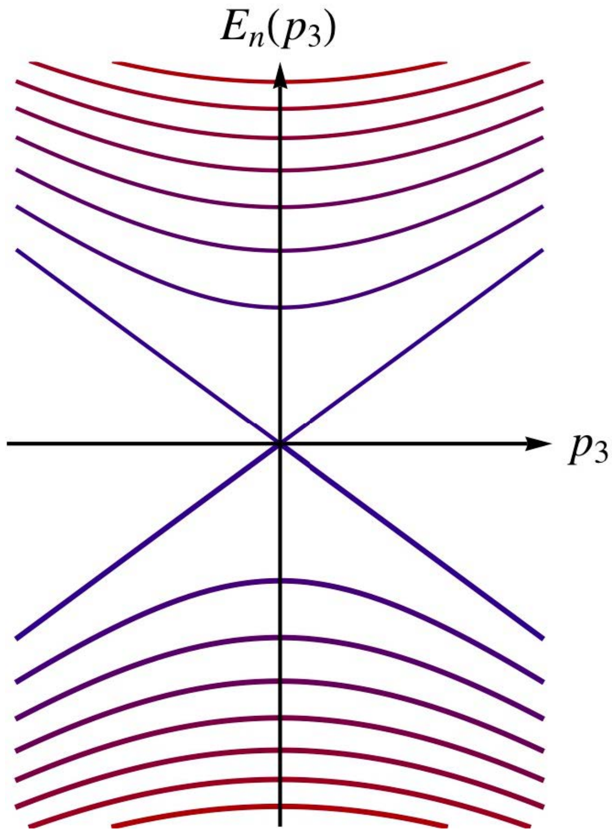
where $s_3 = \pm\frac{1}{2}$ and $n = 0, 1, 2, ...(\text{orbital})$

- An assumption of free quark in 3-direction while $B \to \infty$

$$E_0^{3+1}(p_3) = \pm|p_3|$$

- Propagator looks moving in $1 + 1$ dimension

$$\$(p) \approx i\mathrm{e}^{-\frac{p_\perp^2}{|eB|}} \frac{p_0\gamma_0 - p_3\gamma_3 + m}{p_0^2 - p_3^2 - m^2}(1 - i\gamma^1\gamma^2\mathrm{sgn}(eB))$$



Fermi Propagator at Constant Magnetic fields

- Integral Representative c.f. (J. Schwinger, Phys. Rev. 82, 664 (1951))

$$\not{S}(k) = \not{D}(eB,k)\int_0^\infty \mathrm{d}s\ \exp\left[\mathrm{i}s\left(k_{\shortparallel}^2 - m^2 - k_\perp^2\frac{\tan(eBs)}{eBs}\right)\right]$$

where $\not{D}(eB,k) = (\not{k}_{\shortparallel} + m)\left(1 + \gamma^1\gamma^2\tan(eBs)\right) - \not{k}_\perp\left(1 + \tan^2(eBs)\right)$.

- Summation Representative c.f. (A. Chodos and K. Everding, Phys. Rev. D 42, 2881 (1990))

$$\not{S}(k) = i\exp\left(-\frac{k_\perp^2}{eB}\right)\sum_{n=0}^{\infty}(-1)^n\frac{\not{D}_n(eB,k)}{k_0^2 - k_3^2 - M^2 - 2neB},\quad \mathcal{O}^\pm = \frac{1\pm \mathrm{i}\gamma^1\gamma^2}{2}$$

where $\not{D}_n(eB,k) = 2\not{k}_{\shortparallel}\mathcal{O}^- L_n(\frac{2k_\perp^2}{eB}) - 2\not{k}_{\shortparallel}\mathcal{O}^+ L_{n-1}(\frac{2k_\perp^2}{eB}) + 4\not{k}_\perp L_{n-1}^1(\frac{2k_\perp^2}{eB})$.

- Laguerre Functions c.f. (A. Jeffrey and D. Zwillinger, Table of Integrals, Series, and Products)

$$\sum_{n=0}^{\infty} t^n L_{n-\beta}^\alpha(x) = \frac{t^\beta}{(1-t)^{\alpha+1}}\exp\left[\frac{-tx}{1-t}\right],\quad |t|<1 \text{ and } L_m^\alpha = 0 \text{ if } m<0.$$

The Statement of Dimension Reduction

c.f. (V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Nucl.Phys. B 462, 249 (1996))

$$\begin{aligned}
\langle 0|\bar{\psi}\psi|0\rangle &= -\lim_{x\to y} \mathrm{tr}\, S(x,y) = -\frac{i}{(2\pi)^4}\mathrm{tr} \int d^4k\, \tilde{S}_E(k) \\
&= -\frac{4m}{(2\pi)^4}\int d^4k \int_{1/\Lambda^2}^{\infty} ds \exp\left[-s\left(m^2 + k_4^2 + k_3^2 + \mathbf{k}_\perp^2 \frac{\tanh(eBs)}{eBs}\right)\right] \\
&= -|eB|\frac{m}{4\pi^2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-sm^2} \coth(|eBs|) \\
&\xrightarrow[m\to 0]{} -|eB|\frac{m}{4\pi^2}\left(\ln\frac{\Lambda^2}{m^2} + O(m^0)\right), \qquad (19)
\end{aligned}$$

where $\tilde{S}_E(k)$ is the image of $\tilde{S}(k)$ in Euclidean space and Λ is an ultraviolet cutoff. As is clear from Eqs. (16) and (18), the logarithmic singularity in the condensate appears due to the LLL dynamics.

Quadratic Divergence

However:

$$\coth(x) = \frac{1+\mathrm{e}^{-2x}}{1-\mathrm{e}^{-2x}} \overset{x\to 0}{=} \frac{1}{x}$$

and

$$\begin{aligned} \frac{m|eB|}{4\pi^2}\int_{\frac{1}{\Lambda^2}}^{\infty}\frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\coth(eBs) \quad &\sim \quad \frac{m}{4\pi^2}\int_{\frac{1}{\Lambda^2}}^{\infty}\frac{\mathrm{d}s}{s^2}\,\mathrm{e}^{-sm^2} \\ &\propto \quad \Lambda^2 \end{aligned}$$

It is a quadratic divergence as general Fermi field moving in $d = 4$, not $d = 2$.

Logarithmic Divergence in $d = 2$

Recalculate the chiral condensate in $d = 2$,

$$\begin{aligned}\langle 0|\bar{\psi}\psi|0\rangle &\propto \int \mathrm{d}^2 k \int_{\frac{1}{\Lambda^2}} \mathrm{d}s \exp\left[-sm^2 - k^2 \frac{\tanh(eBs)}{eB}\right] \\ &= (eB) \int_{\frac{1}{\Lambda^2}} \mathrm{d}s \, \frac{\mathrm{e}^{-sm^2}}{\tanh(eBs)} \\ &\sim \ln \frac{\Lambda^2}{m^2}\end{aligned}$$

The native power counting behavior of Fermi field at constant magnetic fields is determinant by dimension and same as without B.

Dimension Regularization

The one loop effective potential is

$$\begin{aligned}
\Omega_{eff} &\propto \int \mathrm{d}s \int \mathrm{d}^d p \exp\left[-s(m^2 + p_{\shortparallel}^2 + p_\perp^2 \frac{\tanh(eBs)}{eBs})\right] \\
&= (eB)\Gamma\left(\frac{d}{2} - 1\right) \int \mathrm{d}s \; s^{1-\frac{d}{2}} \mathrm{e}^{-sm^2} \coth(eBs) \\
&= (eB)\Gamma\left(\frac{d}{2} - 1\right) \sum_{n=0}^{\infty} \int \mathrm{d}s \; s^{1-\frac{d}{2}} \mathrm{e}^{-s(2neB+m^2)} (1 + \mathrm{e}^{-2eBs}) \\
&\sim (eB)\Gamma\left(\frac{d}{2} - 1\right) \Gamma\left(2 - \frac{d}{2}\right) \sum_{n=0}^{\infty} \left(2neB + m^2\right)^{\frac{d}{2}-2}
\end{aligned}$$

which has pole locate in $d = 2, 4, 6, ...$ We have used the geometric summation:

$$\frac{1}{1 - \mathrm{e}^{-x}} = 1 + \mathrm{e}^{-x} + \mathrm{e}^{-2x} + ...$$

Casimir Effect

It is well known for Physicist and Mathematician, that

$$\sum_{n=0}^{\infty} 1 = -\frac{1}{2}, \qquad \sum_{n=0}^{\infty} n = -\frac{1}{12}, \qquad \sum_{n=0}^{\infty} n^2 = 0.$$

the negative sign in the second term explain the Casimir Effect perfectly, because vacuum energy $E_0 = \frac{\pi\hbar c}{L}\sum_{n=0}^{\infty} n$.
c.f. (A. Lambrecht, Physics World, September 2002.)

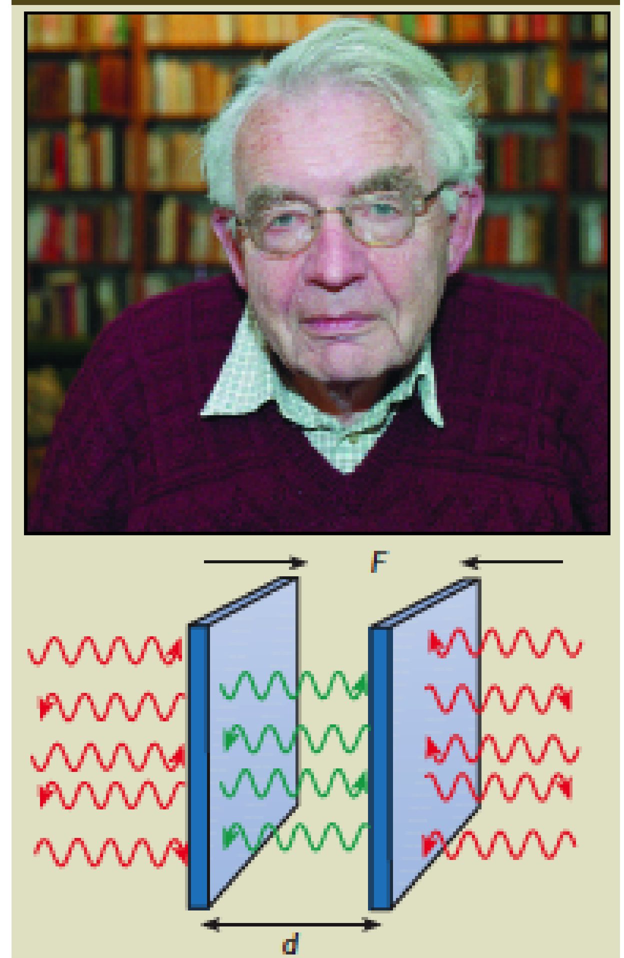
FEATURES

The attractive force between two surfaces in a vacuum – first predicted by Hendrik Casimir over 50 years ago – could affect everything from micromachines to unified theories of nature

The Casimir effect: a force from nothing

Astrid Lambrecht

1 The Casimir force



Regularizing Divergent Series

Or, generally speaking, c.f. (A. Jeffrey and D. Zwillinger, Table of Integrals, Series, and Products)

$$\sum_{n=0}^{\infty} \frac{1}{(n+v)^s} = \zeta(s,v)$$

where

$$\zeta(0,0) = -\frac{1}{2}, \quad \zeta(-1,0) = -\frac{1}{12}, \quad \zeta(-2,0) = 0.$$

Riemann Zeta function regularization is equivalent to the Zero Point Energy subtraction procedure of the Casimir effect.

Tip: Zeta and its related functions are natural candidate to deal with the summation of all Landau levels and remove the associated infinities.

Back to $d = 4 - \epsilon$

$$\begin{aligned}
\Gamma\left(\frac{\epsilon}{2}\right)\sum_{n=0}^{\infty}\left(2neB+m^2\right)^{\frac{4-\epsilon}{2}-2} &= \sum_{n=0}^{\infty}(-\gamma-\log(2eB)) - \frac{\partial}{\partial\alpha}\sum_{n=0}^{\infty}\left(n+\frac{m^2}{2eB}\right)^{\alpha}\Big|_{\alpha=0} \\
&= \frac{1}{2}(\gamma+\log(2eB)) - \zeta'\left(0,\frac{m^2}{2eB}\right) \\
&= \frac{1}{2}(\gamma+\log(2eB)+\log(2\pi)) - \log\Gamma\left(\frac{m^2}{2eB}\right)
\end{aligned}$$

$$\Omega_{eff} = \begin{cases} m^2\log m^2 & B\to 0; \\ eB\log(eB) & B\to\infty. \end{cases}$$

Regulator $\epsilon_\perp$

An example of the regulator ϵ to sum infinity series:

$$\sum_{n=0}^{\infty} n = \sum_{n=0}^{\infty} \frac{\partial}{\partial \epsilon} \mathrm{e}^{\epsilon n} = \frac{\partial}{\partial \epsilon} \frac{1}{1-\mathrm{e}^{\epsilon}} = \frac{\mathrm{e}^{\epsilon}}{(1-\mathrm{e}^{\epsilon})^2}$$

In[1]:= $\mathbf{Series}\left[\dfrac{e^{\epsilon}}{(1-e^{\epsilon})^2}, \{\epsilon, 0, 1\}\right]$

Out[1]= $\dfrac{1}{\epsilon^2} - \dfrac{1}{12} + O(\epsilon^2)$

Note: to distinguish the ϵ which have used in dimension regularization, we employ $\epsilon_\perp$ to denote summation regulator in the future.

Lerch-Zeta Function

c.f. (A. Jeffrey and D. Zwillinger, Table of Integrals, Series, and Products)

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int \mathrm{d}\tau \; \frac{\tau^{s-1} \mathrm{e}^{-v\tau}}{1 - z\mathrm{e}^{-\tau}}$$

for $\mathrm{Re}\,(s) > 0 \;\cap\; |z| \leq 1 \;\cap\; z \neq 1; \mathrm{Re}\,(s) > 1 \;\cap\; z = 1.$

Therefore, at $d = 4 - \epsilon$, under the help of regulator $\mathrm{e}^{-\epsilon_\perp} = z$,

$$\begin{aligned}
\Omega_{eff} &= (eB)\Gamma\left(\frac{d}{2} - 1\right) \int \mathrm{d}s \; s^{1-\frac{d}{2}} \mathrm{e}^{-sm^2} \coth(eBs) \\
&= (eB)^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2} - 1\right) \int \mathrm{d}\tau \; \tau^{1-\frac{d}{2}} \frac{1 + \mathrm{e}^{-2\tau}}{1 - z\mathrm{e}^{-2\tau}} \mathrm{e}^{-\tau\frac{m^2}{2eB}} \\
&= (eB)^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2} - 1\right) \Gamma\left(\frac{\epsilon}{2}\right) \left(\Phi\left(z, \frac{\epsilon}{2}, \frac{m^2}{2eB}\right) + \Phi\left(z, \frac{\epsilon}{2}, \frac{m^2}{2eB} + 1\right) \right)
\end{aligned}$$

where $\tau = eBs$.

Properties of Lerch-Zeta Function

c.f. (A. Jeffrey and D. Zwillinger, Table of Integrals, Series, and Products)

- Limiting behaviors

$$\lim_{z \to 1} \Phi(z, s, v) = \frac{\Gamma(1-s)}{(1-z)^{1-s}} + \zeta(s, v) \quad \text{for } \mathrm{Re}\,(s) < 1;$$

$$\lim_{z \to 1} \Phi(z, 1, v) = -\log(1-z) - \psi(v) \quad \text{for } s = 1.$$

- Derivatives

$$D_z \Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z \mathrm{e}^{-\tau} \tau^{s-1} \exp(-v\tau)}{(1 - z\mathrm{e}^{-\tau})^2} \, \mathrm{d}\tau$$

$$D_z \Phi(z, s, v) = \Phi(z, s-1, v) - v\Phi(z, s, v)$$

Effective Potential, again

Physicists are always bold to discard the infinites. But, the left

$$\begin{aligned} \Omega_{eff} &\propto \lim_{z\to 1} \Gamma\left(\frac{\epsilon}{2}\right) \Phi\left(z, \frac{\epsilon}{2}, \frac{m^2}{2eB}\right) \\ &= \frac{2}{\epsilon^2} - \frac{\gamma}{\epsilon} - \gamma\zeta\left(0, \frac{m^2}{2eB}\right) + \zeta'\left(0, \frac{m^2}{2eB}\right) \end{aligned}$$

Different ways to obtain the same accurate and quantitative results.

Dimension Reduction of Neutral Fields

Before, based on the statement of dimension reduction of fermi field at strong magnetic fields limit. It is general accepted that the neutral fields affect deeply by the Lowest Landau Level Fermi fields.

c.f. (K. Fukushima and Y. Hidaka, Phys. Rev. Lett. 110, 031601 (2013))

Magnetic Catalysis vs Magnetic Inhibition

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We discuss the fate of chiral symmetry in an extremely strong magnetic field B. We investigate not only quark fluctuations but also neutral meson effects. The former would enhance the chiral-symmetry breaking at finite B according to the Magnetic Catalysis, while the latter would suppress the chiral condensate once B exceeds the scale of the hadron structure. Using a chiral model we demonstrate how neutral mesons are subject to the dimensional reduction and the low dimensionality favors the chiral-symmetric phase. We point out that this effect, the Magnetic Inhibition, can be a feasible explanation for recent lattice-QCD data indicating the decreasing behavior of the chiral-restoration temperature with increasing B.

PACS numbers: 11.30.Rd, 21.65.Qr, 12.38.-t

Photon Propagator in the Strong Field

c.f. (B. Jancovici, Phys. Rev. 187, 2275 (1969); G. Calucci and R. Ragazzon, J. Phys. A 27, 2161 (1994); V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. 83, 1291 (1999))

$$
\begin{aligned}
\pi_{\shortparallel}(k_{\shortparallel}^2) &\propto \frac{eB}{m^2}, \quad \text{as } k_{\shortparallel}^2 \ll m^2; \\
\pi_{\shortparallel}(k_{\shortparallel}^2) &\propto \frac{eB}{k_{\shortparallel}^2}, \quad \text{as } k_{\shortparallel}^2 \gg m^2; \\
\pi_{\perp}(k_{\shortparallel}^2, k_{\perp}^2) &= 0.
\end{aligned}
$$

3. In the one-loop approximation, with fermions from the LLL, the photon propagator takes the following form in covariant gauges [11,3]:

$$
\mathcal{D}_{\mu\nu}(k) = -i\left[\frac{1}{k^2}g^{\perp}_{\mu\nu} + \frac{k^{\parallel}_{\mu}k^{\parallel}_{\nu}}{k^2 k^2_{\parallel}} + \frac{1}{k^2 + k^2_{\parallel}\Pi(k^2_{\perp},k^2_{\parallel})} \times \left(g^{\parallel}_{\mu\nu} - \frac{k^{\parallel}_{\mu}k^{\parallel}_{\nu}}{k^2_{\parallel}}\right) - \frac{\lambda}{k^2}\frac{k_{\mu}k_{\nu}}{k^2}\right], \qquad (12)
$$

where λ is a gauge parameter. The explicit expression for $\Pi(k^2_{\perp}, k^2_{\parallel}) = \exp[-(k_{\perp}l)^2/2]\Pi(k^2_{\parallel})$ is given in Refs. [11,3]. For our purposes, it is sufficient to know its asymptotes,

$$
\Pi(k^2_{\parallel}) \simeq \frac{\bar{\alpha}}{3\pi}\frac{|eB|}{m^2} \quad \text{as} \quad |k^2_{\parallel}| \ll m^2, \qquad (13)
$$

$$
\Pi(k^2_{\parallel}) \simeq -\frac{2\bar{\alpha}}{\pi}\frac{|eB|}{k^2_{\parallel}} \quad \text{as} \quad |k^2_{\parallel}| \gg m^2, \qquad (14)
$$

where $\bar{\alpha} = N\alpha$. The polarization effects are absent in the transverse components of $\mathcal{D}_{\mu\nu}(k)$. This is because the bare vertex for fermions from the LLL is $O^{(-)}\gamma^{\mu}O^{(-)} = O^{(-)}\gamma^{\mu}_{\parallel}$. Therefore the LLL fermions couple only to the longitudinal (0,3) components of the photon field. Then, there is a strong screening effect in the $\left(g^{\parallel}_{\mu\nu} - k^{\parallel}_{\mu}k^{\parallel}_{\nu}/k^2_{\parallel}\right)$ component of the photon propagator. For $m^2 \ll |k^2_{\parallel}| \ll |eB|$ and $|k^2_{\perp}| \ll |eB|$, Eq. (14) implies that

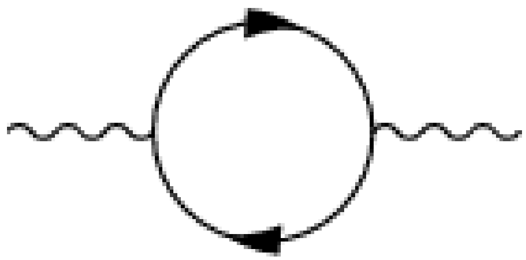
$$
\frac{1}{k^2 + k^2_{\parallel}\Pi(k^2_{\perp},k^2_{\parallel})} \simeq \frac{1}{k^2 - M^2_{\gamma}}, \qquad (15)
$$

with $M^2_{\gamma} = 2\bar{\alpha}|eB|/\pi$. This is reminiscent of the Higgs effect in the (1+1)-dimensional QED (Schwinger model) [12,13].

An Application of Lerch-Zeta Function

$$\Pi^{\mu\nu} = \mathrm{Tr}[\not{S}(k)\gamma^{\mu}\not{S}(p)\gamma^{\nu}]$$

where $p = k + q$.



$$\Pi^{\mu\nu} = \int \mathrm{d}\Gamma \int_0^{\infty} \mathrm{d}\tau \; I^{\mu\nu} \tau \exp\left[-\left(\hat{k}_{\shortparallel}{}^2 - x(1-x)\hat{q}_{\shortparallel}{}^2 + \hat{M}^2 + nx + m(1-x) + i\epsilon\right)\tau\right]$$

where $\hat{M}^2 = M^2/(2eB)$ and $\hat{q}^2 = q^2/(2eB)$. We have added a regulator $z^2 = \mathrm{e}^{-\epsilon_{\perp}}$ in each summation,

$$\mathrm{d}\Gamma = (-1)^{n+m} \sum_{n=0}^{\infty} z^{\frac{n}{2}} \sum_{m=0}^{\infty} z^{\frac{m}{2}} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{2-\epsilon}\hat{k}_{\shortparallel}}{(2\pi)^2} \int \frac{\mathrm{d}^2\hat{k}_{\perp}}{(2\pi)^2} \exp\left[-2\hat{k}_{\perp}^2\right] \exp\left[-2\hat{p}_{\perp}^2\right]$$

$$I^{\mu\nu} = A^{\mu\nu}(L_n L_m + L_{n-1}L_{m-1}) + B^{\mu\nu} L_{n-1} L_m + C^{\mu\nu} L^1_{n-1} L^1_{m-1}$$

An Application of Lerch-Zeta Function, cont

According to the properties of Laguerre polynomials, we are able to transfer the summation of Landau level to a general easy way, and such as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{(n+m)}\mathrm{e}^{-nx\tau}z^{\frac{n}{2}}\mathrm{e}^{-m(1-x)\tau}z^{\frac{m}{2}}L_n(4\hat{k_\perp}^2)L_m(4\hat{p_\perp}^2)$$

$$=\frac{1}{1-t_1}\exp\left[\frac{-4t_1\hat{k_\perp}^2}{1-t_1}\right]\frac{1}{1-t_2}\exp\left[\frac{-4t_2\hat{p_\perp}^2}{1-t_2}\right]$$

where $t_1=-\sqrt{z}\mathrm{e}^{-x\tau}$, $t_2=-\sqrt{z}\mathrm{e}^{-(1-x)\tau}$.

$$\Pi^{\mu\nu} = \frac{1}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^\infty \mathrm{d}\tau\frac{\tau^{\frac{\epsilon}{2}}\mathrm{e}^{-\left(\hat{M}^2-x(1-x)\hat{q}^2+i\epsilon\right)\tau}}{4(1-z\mathrm{e}^{-\tau})}\pi^{\mu\nu}$$

All information transfer to the knowledge of Lerch-Zeta function.

Details of Tensor Structure

$$\pi^{\mu\nu} = \left(-\epsilon\frac{g_{\shortparallel}^{\mu\nu}}{\tau} - x(1-x)\left(4q_{\shortparallel}^{\mu}q_{\shortparallel}^{\nu} - 2g_{\shortparallel}^{\mu\nu}q_{\shortparallel}^{2}\right) + 2g_{\shortparallel}^{\mu\nu}M^2\right)\left(1+\mathrm{e}^{-\tau}\right)$$
$$+\left(-2\epsilon g_{\perp}^{\mu\nu} + 4g_{\shortparallel}^{\mu\nu}\right)\frac{\mathrm{e}^{-\tau}}{1-z\mathrm{e}^{-\tau}} + g_{\perp}^{\mu\nu}\left(\frac{4}{\tau} + 4x(1-x)q_{\shortparallel}^{2} + 4M^2\right)\mathrm{e}^{-(1-x)\tau}$$
$$-4x(1-x)\left(2q_{\perp}^{\mu}q_{\perp}^{\nu} + 2q_{\shortparallel}^{\mu}q_{\perp}^{\nu} + 2q_{\perp}^{\mu}q_{\shortparallel}^{\nu} - g^{\mu\nu}q_{\perp}^{2}\right)\frac{\tau\mathrm{e}^{-\tau}}{1-z\mathrm{e}^{-\tau}}$$

$$\Pi^{\mu\nu} = P_{\shortparallel}^{\mu\nu}(q^2)\pi_{\shortparallel} + P_{\perp}^{\mu\nu}(q^2)\pi_{\perp}$$

where $P_{\shortparallel}^{\mu\nu}(q^2) = q_{\shortparallel}^{\mu}q_{\shortparallel}^{\nu} - g_{\shortparallel}^{\mu\nu}q_{\shortparallel}^{2}$ and $P_{\perp}^{\mu\nu}(q^2) = q^{\mu}q^{\nu} - g^{\mu\nu}q^2 - P_{\shortparallel}^{\mu\nu}(q^2)$.

Interpretation of π

$$\begin{aligned}
\pi_{\shortparallel} &= \frac{1}{4\pi^2}\int_0^1 x(1-x)\lim_{z\to 1}\left[\Phi(z,1,v)+\Phi(z,1,v+1)\right]\mathrm{d}x \\
&= -\frac{1}{4\pi^2}\int_0^1 x(1-x)\left[\psi(v)+\psi(v+1)\right]\mathrm{d}x \\
\pi_{\perp} &= \frac{1}{2\pi^2}\int_0^1 x(1-x)\lim_{z\to 1} D_z\Phi(z,2,v)\mathrm{d}x \\
&= -\frac{1}{2\pi^2}\int_0^1 x(1-x)\left[\psi(v)+v\zeta(2,v)\right]\mathrm{d}x
\end{aligned}$$

$$\mathbf{v}=\frac{\mathbf{M^2-x(1-x)q^2}}{\mathbf{2eB}}$$

Because $\psi(x)\to\ln x$ for $x\to\infty$. $\pi_\perp$ and $\pi_{\shortparallel}$ are real and analytic. They logarithmically divergent if $v\to\infty$, i.e. when $M^2\gg q^2$ & $M^2\gg eB$. Such logarithmic divergence is same as the vacuum polarization without B. A counter term has to be included:

$$\hat{\pi}_{\shortparallel,\perp}(B,q^2)=\pi^{bare}_{\shortparallel,\perp}(B,q^2)-\pi_{\shortparallel,\perp}(B,q^2=0)$$

Threshold of Pair Production

In the large momentum limit, beginning at $q^2 = 4M^2$, it yields a kinematic condition to create an imaginary part of $\pi_{\shortparallel}(q^2)$, while as the threshold of $\pi_\perp$ is $q^2 = 4(M^2 + 2eB)$. Similar result also has obtained in c.f. (K. Hattori and K. Itakura, Ann. Phys. 330, 23 (2013); F. Karbstein, Phys. Rev. D 88, 085033 (2013).K. Ishikawa, D. Kimura, K. Shigaki and A. Tsuji, Int. J. Mod. Phys. A 28, 1350100 (2013).)

When $v < 0$ and/or $v + 1 < 0$, Di-gamma function ψ has simple poles at non-positive integer. Let

$$N = \left\lfloor \frac{\frac{1}{4}q^2 - M^2}{2eB} \right\rfloor_{floor}$$

The quantum states of virtual particles are occupied up to the N-th Landau level! N is at least equal to 0 in the chiral limit $M \to 0$. Also:

$$\mathrm{Im}\,\pi_{\shortparallel,\perp}(q^2) \propto \frac{\pi}{3}\left(\sqrt{1 - \frac{4M^2}{q^2}}\left(1 + \frac{2M^2}{q^2}\right) + \sqrt{1 - \frac{4M^2 + 8eB}{q^2}}\left(1 + \frac{2M^2 + 4eB}{q^2}\right)\right)$$

Results, Some Out of Expectations

- The transverse part of vacuum polarization is not damped.
- Transverse and longitudinal parts are closing to uniform in the weak B limit.
- The vacuum polarization is pure real if below the energy threshold $q^2 < 4M^2$.
- Beyond the threshold, the imaginary part is the dominant contribution of $\pi(q^2)$.
- The imaginary part always exists and is important in the chiral limit.
- The LLL approximation is only valid for massive Fermions and the results are only correct in the region where $eB \gg M^2 \gg q^2$.

Conclusions

- Magnetic fields open a new window in the study of QCD matter at extreme condition.
- It is very dangerous to cut off the Landau levels to several ones.
- Lerch-Zeta function is powerful to regularize the summation of infinity series.
- Chiral limit, $M \to 0$ and strong field limit, $B \to \infty$ are non-communication.
- Recalculating the gap equation with full Landau levels consideration at external magnetic fields is necessary. c.f. (Lang Yu, in preparation)

Thank You for Your Attention !